

# ON THE CHARACTERIZATION OF HONEST TIMES AVOIDING ALL STOPPING TIMES

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**ABSTRACT.** We present a self-contained proof of the following result: a random time is an honest time avoiding all stopping times if and only if it coincides with the (first) time that a nonnegative local martingale with continuous supremum process and zero terminal value achieves its overall maximum, given that this time of maximum is almost surely finite. The requirement that all martingales on the stochastic basis have continuous paths, which was present in previous literature, is dropped.

## 1. THE CHARACTERIZATION RESULT

Consider a filtered probability space  $(\Omega, \mathbf{F}, \mathbb{P})$ , where  $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$  is a filtration satisfying the usual conditions of right-continuity and saturation by  $\mathbb{P}$ -null sets of  $\mathcal{F} := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t$ . All (local) martingales and supermartingales on  $(\Omega, \mathbf{F}, \mathbb{P})$  are assumed to have càdlàg paths.

A *random time* is a  $[0, \infty]$ -valued,  $\mathcal{F}$ -measurable random variable. The random time  $\rho$  is said to *avoid all stopping times* if  $\mathbb{P}[\rho = \tau] = 0$  holds whenever  $\tau$  is a (possibly, infinite-valued) stopping time. The random time  $\rho$  is an *honest time* if for all  $t \in \mathbb{R}_+$  there exists an  $\mathcal{F}_t$ -measurable random variable  $R_t$  such that  $\rho = R_t$  holds on  $\{\rho \leq t\}$ . Honest times constitute the most important class of random times outside the realm of stopping times. They have been extensively studied in the literature, especially in relation to filtration enlargements—see, for example, the original paper [1]. It is impossible to present here the vast literature on the subject of honest times; we single out the monograph [2], simply because reference to it will be made in the proof of Theorem 1.1 below.

Use  $\mathcal{M}_0^\uparrow$  to denote the set of all nonnegative local martingales  $L$  with  $L_0 = 1$ ,  $L_\infty := \lim_{t \rightarrow \infty} L_t = 0$ , and such that its supremum process  $L^* := \sup_{t \in [0, \cdot]} L_t$  is continuous, all the previous holding in the  $\mathbb{P}$ -a.s. sense. For  $L \in \mathcal{M}_0^\uparrow$ , define

$$(1.1) \quad \rho_{\min}^L := \inf \{t \in \mathbb{R}_+ \mid L_t = L_\infty^*\},$$

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where by convention we set  $\rho_{\min}^L = \infty$  if the latter set is empty. Since  $L$  is càdlàg,  $L_{\rho_{\min}^L} = L_{\infty}^*$  holds on the event  $\{\rho_{\min}^L < \infty\}$ .

Use  $\mathcal{L}_0^\uparrow$  to denote the set of all  $L \in \mathcal{M}_0^\uparrow$  such that additionally  $\mathbb{P}[\rho_{\min}^L < \infty] = 1$  holds. For  $L \in \mathcal{L}_0^\uparrow$  and  $t \in \mathbb{R}_+$ , define the  $\mathcal{F}_t$ -measurable random variable  $R_t^L := \inf\{s \in [0, t] \mid L_s = L_s^*\} \wedge t$ , and note that  $\rho_{\min}^L = R_t^L$  holds on  $\{\rho_{\min}^L \leq t\}$ . Therefore,  $\rho_{\min}^L$  is an honest time. When  $L \in \mathcal{L}_0^\uparrow$ , it is also true that  $\rho_{\min}^L$  avoids all stopping times, and that it is actually the canonical example of an honest time avoiding all stopping times, as the following result shows.

**Theorem 1.1.** *For a random time  $\rho$ , the following two statements are equivalent:*

- (1)  $\rho$  is an honest time avoiding all stopping times.
- (2)  $\rho = \rho_{\min}^L$  holds for some  $L \in \mathcal{L}_0^\uparrow$ .

Under the additional assumption that all martingales on  $(\Omega, \mathbf{F}, \mathbb{P})$  have  $\mathbb{P}$ -a.s. continuous paths, a version of Theorem 1.1 was established in [3]. To the best of the author's knowledge, a proof of Theorem 1.1 without the aforementioned assumption has not appeared previously in the literature.

*Remark 1.2.* Note that, instead of  $\rho_{\min}^L$  of (1.1), the *last* time of supremum of  $L$  is used in [3] via considering  $\rho_{\max}^L := \sup\{t \in \mathbb{R}_+ \mid L_t = L_\infty^*\}$ . Furthermore, the requirement that the corresponding random time is  $\mathbb{P}$ -a.s. finite is thereby absent. In that setting, the  $\mathbb{P}$ -a.s. path-continuity of  $L$  automatically implies both that  $\mathbb{P}[\rho_{\max}^L < \infty] = 1$  and  $L$  sampled at  $\rho_{\max}^L$  actually achieves the maximum when  $L \in \mathcal{L}_0^\uparrow$ . In contrast, in the case where jumps may be present the extra requirement  $\mathbb{P}[\rho_{\min}^L < \infty] = 1$  for  $L \in \mathcal{M}_0^\uparrow$  to belong to  $\mathcal{L}_0^\uparrow$  is essential for the validity of Theorem 1.1. Indeed, consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  that supports a  $\mathbb{R}_+$ -valued random variable  $\tau$  such that  $\mathbb{P}[\tau > t] = e^{-t}$  holds for all  $t \in \mathbb{R}_+$ . Define  $\mathbf{F}$  as the (usual augmentation of the) smallest filtration which makes  $\tau$  a stopping time. Then, define the process  $L$  via  $L_t = \exp(t)\mathbb{I}_{\{\tau > t\}}$  for all  $t \in \mathbb{R}_+$ . It is straightforward to check that  $L \in \mathcal{M}_0^\uparrow$ . However, since  $\mathbb{P}[L_\tau = 0] = 1$ , it is immediate that  $\mathbb{P}[\rho_{\min}^L = \infty] = 1$ , which shows in a dramatic fashion the failure of  $\rho_{\min}^L$  to avoid all stopping times.

*Remark 1.3.* Whenever  $L \in \mathcal{L}_0^\uparrow$  and  $\rho$  is *any* random time such that  $\mathbb{P}[L_\rho = L_\infty^*] = 1$  holds, Lemma 2.2 in Section 2 implies that  $\mathbb{P}[\rho = \rho_{\min}^L] = 1$ . Therefore, a process  $L \in \mathcal{L}_0^\uparrow$  has a  $\mathbb{P}$ -a.s. unique time of maximum.

## 2. PROOF OF THEOREM 1.1

During the course of the proof of Theorem 1.1, and in an effort to be as self-contained as possible, we shall provide details for every step.

For a random time  $\sigma$  and a process  $X = (X_t)_{t \in \mathbb{R}_+}$ ,  $X^\sigma = (X_{\sigma \wedge t})_{t \in \mathbb{R}_+}$  will denote throughout the process  $X$  stopped at  $\sigma$ . For any unexplained, but fairly standard, notation and facts regarding stochastic analysis, we refer the reader to [4].

**A couple of auxiliary results.** The two results presented below will be used in the proof of both implications later. The first auxiliary result is a slightly elaborate version of Doob's maximal identity—see [3]. It will be useful throughout, sometimes in its “conditional” version. The second result implies in particular that  $\rho_{\min}^L$  avoids all stopping times whenever  $L \in \mathcal{L}_0^\uparrow$ .

**Lemma 2.1.** *Let  $L$  be a nonnegative local martingale with  $L_0 = 1$ . Then,  $\mathbb{P}[L_\infty^* > x] \leq 1/x$  holds for all  $x \in (1, \infty)$ . Furthermore,  $\mathbb{P}[L_\infty^* > x] = 1/x$  holds for all  $x \in (1, \infty)$  if and only if  $L \in \mathcal{M}_0^\uparrow$ .*

*Proof.* For  $x \in (1, \infty)$ , define the stopping time  $\tau_x := \inf\{t \in \mathbb{R}_+ \mid L_t > x\}$ , and note that  $\{L_\infty^* > x\} = \{\tau_x < \infty\}$ . Since  $\mathbb{E}[L_{\tau_x}^*] \leq x + \mathbb{E}[L_{\tau_x}] \leq x + 1$ ,  $L^x$  is a uniformly integrable martingale for all  $x \in (1, \infty)$ . It follows that  $x\mathbb{P}[L_\infty^* > x] = x\mathbb{P}[\tau_x < \infty] = \mathbb{E}[x\mathbb{I}_{\{\tau_x < \infty\}}] \leq \mathbb{E}[L_{\tau_x}] = 1$  for  $x \in (1, \infty)$ , with equality holding if and only if  $\mathbb{P}[L_{\tau_x} = x\mathbb{I}_{\{\tau_x < \infty\}}] = 1$ . Whenever  $L \in \mathcal{M}_0^\uparrow$ , the equality  $\mathbb{P}[L_{\tau_x} = x\mathbb{I}_{\{\tau_x < \infty\}}] = 1$  is immediate for all  $x \in (1, \infty)$ . Conversely, assume that  $\mathbb{P}[L_{\tau_x} = x\mathbb{I}_{\{\tau_x < \infty\}}] = 1$  holds for all  $x \in (1, \infty)$ . It is clear that  $L^*$  must have  $\mathbb{P}$ -a.s. continuous paths; furthermore, since  $\mathbb{P}[\bigcup_{n \in \mathbb{N}} \{\tau_n = \infty\}] = 1$ ,  $\mathbb{P}[L_\infty = 0] = 1$  follows. Therefore,  $L \in \mathcal{M}_0^\uparrow$ .  $\square$

**Lemma 2.2.** *Suppose that  $L \in \mathcal{M}_0^\uparrow$ , and let  $\rho$  be any random time such that  $\mathbb{P}[L_\rho = L_\infty^*] = 1$ . Then,  $L \in \mathcal{L}_0^\uparrow$ ,  $\mathbb{P}[\rho = \rho_{\min}^L] = 1$ , and  $\rho$  avoids all stopping times.*

*Proof.* Since  $\mathbb{P}[\rho = \infty, L_\rho = L_\infty^*] = 0$  holds for  $L \in \mathcal{M}_0^\uparrow$ ,  $\mathbb{P}[\rho = \infty] = 0$  follows. In view of the obvious equality  $\mathbb{P}[\rho_{\min}^L \leq \rho] = 1$ , we obtain  $L \in \mathcal{L}_0^\uparrow$ . Note the following set-inclusions for each  $t \in \mathbb{R}_+$ , valid modulo  $\mathbb{P}$ :

$$\left\{ \sup_{v \in [t, \infty)} L_v > L_t^* \right\} \subseteq \{\rho_{\min}^L > t\}, \quad \{\rho > t\} \subseteq \left\{ \sup_{v \in [t, \infty)} L_v \geq L_t^* \right\}.$$

(The fact that  $\mathbb{P}[\rho < \infty]$  is used in the second set-inequality above.) A use of a conditional version of Lemma 2.1 gives

$$\mathbb{P} \left[ \sup_{v \in [t, \infty)} L_v \geq L_t^* \mid \mathcal{F}_t \right] = \frac{L_t}{L_t^*} = \mathbb{P} \left[ \sup_{v \in [t, \infty)} L_v > L_t^* \mid \mathcal{F}_t \right], \quad \text{for } t \in \mathbb{R}_+.$$

It follows that  $\mathbb{P}[\rho > t] \leq \mathbb{P}[\rho_{\min}^L > t]$  holds for all  $t \in \mathbb{R}_+$ . Combined with  $\mathbb{P}[\rho_{\min}^L \leq \rho < \infty] = 1$ , it follows that  $\mathbb{P}[\rho = \rho_{\min}^L] = 1$ .

Given  $\mathbb{P}[\rho = \rho_{\min}^L] = 1$ , for  $\rho$  to avoid all stopping times, it suffices that  $\rho_{\min}^L$  does so. Pick some stopping time  $\tau$ ; it will be shown in the sequel that  $\mathbb{P}[\rho_{\min}^L = \tau \mid \mathcal{F}_\tau] = 0$  holds. Indeed, on  $\{\tau = \infty\} \cup \{\tau < \infty, L_\tau < L_\tau^*\}$ ,  $\mathbb{P}[\rho_{\min}^L = \tau \mid \mathcal{F}_\tau] = 0$  trivially holds. (Recall that  $\mathbb{P}[\rho_{\min}^L = \infty] = 0$ .) On  $\{\tau < \infty, L_\tau = L_\tau^*\}$ , where in particular  $L_\tau > 0$ , a conditional form of Lemma 2.1 gives that  $\mathbb{P}[\sup_{t \in [\tau, \infty)} L_t > L_\tau^* \mid \mathcal{F}_\tau] = L_\tau/L_\tau^* = 1$  holds; therefore,  $\mathbb{P}[\rho_{\min}^L = \tau \mid \mathcal{F}_\tau] = 0$ .  $\square$

**2.1. Proof of implication (2)  $\Rightarrow$  (1).** It has already been established that  $\rho_{\min}^L$  is an honest time if  $L \in \mathcal{L}_0^\uparrow$ . Implication (2)  $\Rightarrow$  (1) then follow immediately from Lemma 2.2, which in particular implies that  $\rho_{\min}^L$  avoids all stopping times whenever  $L \in \mathcal{L}_0^\uparrow$ .

**2.2. Proof of implication (1)  $\Rightarrow$  (2).** Throughout the proof of implication (1)  $\Rightarrow$  (2), fix an honest time  $\rho$  avoiding all stopping times. Let  $Z$  be the nonnegative càdlàg Azéma supermartingale (see [2] and the references therein) that satisfies  $Z_t = \mathbb{P}[\rho > t \mid \mathcal{F}_t]$  for all  $t \in \mathbb{R}_+$ . The next result follows from [2, Lemma 4.3(i) and Proposition 5.1]—we provide its proof for completeness.

**Lemma 2.3.** *Suppose that  $\rho$  is an honest time avoiding all stopping times. Then,  $\mathbb{P}[Z_\rho = 1] = 1$ .*

*Proof.* Let  $(R_t^0)_{t \in \mathbb{R}_+}$  be an adapted process such that  $\rho = R_t^0$  holds on  $\{\rho \leq t\}$  for all  $t \in \mathbb{R}_+$ . Note that the adapted process  $(R_t^0 \wedge t)_{t \in \mathbb{R}_+}$  has the same property as well; therefore, we may assume that  $R_t^0 \leq t$  holds for all  $t \in \mathbb{R}_+$ . With  $\mathbb{D}$  denoting a dense countable subset of  $\mathbb{R}_+$ , define the process  $R := \lim_{\mathbb{D} \ni t \downarrow} (\sup_{s \in \mathbb{D} \cap (0, t)} R_s^0)$ ; then,  $R$  is right-continuous, adapted and non-decreasing, and  $R_t \leq t$  still holds for all  $t \in \mathbb{R}_+$ . Furthermore, since for  $s \in \mathbb{R}_+$  and  $t \in \mathbb{R}_+$  with  $s \leq t$ ,  $\rho = R_s^0 = R_t^0$  holds on  $\{\rho \leq s\} \subseteq \{\rho \leq t\}$ , it follows that  $\rho = R_t$  holds on  $\{\rho \leq t\}$  for all  $t \in \mathbb{R}_+$ . Define a  $\{0, 1\}$ -valued optional process  $I$  via  $I_t = \mathbb{I}_{\{R_t = t\}}$  for  $t \in \mathbb{R}_+$ . The properties of  $R$  can be seen to imply  $\{I = 1\} \subseteq \llbracket 0, \rho \rrbracket$ , as well as  $I_\rho = 1$  on  $\{\rho < \infty\}$ ; since  $\mathbb{P}[\rho = \infty] = 0$  holds due to the fact that  $\rho$  avoids all stopping times, we conclude that  $\mathbb{P}[I_\rho = 1] = 1$ . Fix a finite stopping time  $\tau$ . Using again the fact that  $\rho$  avoids all stopping times,  $Z_\tau = \mathbb{P}[\rho \geq \tau \mid \mathcal{F}_\tau]$  holds. Then,  $I_\tau \in \mathcal{F}_\tau$  and  $\{I = 1\} \subseteq \llbracket 0, \rho \rrbracket$  imply that  $\mathbb{E}[I_\tau Z_\tau] = \mathbb{E}[I_\tau \mathbb{I}_{\{\tau \leq \rho\}}] = \mathbb{E}[I_\tau]$ . Since  $I$  is  $\{0, 1\}$ -valued and  $Z$  is  $[0, 1]$ -valued,  $\mathbb{E}[I_\tau Z_\tau] = \mathbb{E}[I_\tau]$  implies that  $\{I_\tau = 1\} \subseteq \{Z_\tau = 1\}$ . Since the latter holds for all finite stopping times  $\tau$  and both  $I$  and  $Z$  are optional, the optional section theorem implies that  $\{I = 1\} \subseteq \{Z = 1\}$ , modulo  $\mathbb{P}$ -evanescence. Then,  $\mathbb{P}[I_\rho = 1] = 1$  implies  $\mathbb{P}[Z_\rho = 1] = 1$ .  $\square$

Continuing, let  $A$  be the unique (up to  $\mathbb{P}$ -evanescence) adapted, càdlàg, nonnegative and non-decreasing process such that  $\mathbb{E}_{\mathbb{P}}[V_\rho] = \mathbb{E}_{\mathbb{P}}[\int_0^\infty V_t dA_t]$  holds for all nonnegative optional processes  $V$ —in other words,  $A$  is the dual optional projection of  $\mathbb{I}_{[\rho, \infty]}$ . Note the equality  $\mathbb{E}[A_\tau - A_{\tau-}] = \mathbb{P}[\rho = \tau] = 0$ , holding for all finite stopping times  $\tau$ , which implies by the optional section theorem that  $A_0 = 0$  and  $A$  has  $\mathbb{P}$ -a.s. continuous paths. Then,  $Z = M - A$  is the Doob-Meyer decomposition of  $Z$ , where  $M$  is the nonnegative martingale such that  $M_t = \mathbb{E}_{\mathbb{P}}[A_\infty \mid \mathcal{F}_t]$  holds for all  $t \in \mathbb{R}_+$ . For each  $n \in \mathbb{N}$ , define the stopping time  $\zeta_n := \inf\{t \in \mathbb{R}_+ \mid Z_t < 1/n\}$ , and set  $\zeta := \lim_{n \rightarrow \infty} \zeta_n = \inf\{t \in \mathbb{R}_+ \mid Z_{t-} = 0 \text{ or } Z_t = 0\}$ . Define the  $[0, 1]$ -valued continuous non-decreasing adapted process  $K = 1 - \exp(-\int_0^{\zeta \wedge \cdot} (1/Z_t) dA_t)$ . Since  $A$  has continuous paths,  $\mathbb{P}[K_{\zeta_n} < 1] = 1$  holds for all  $n \in \mathbb{N}$ . Defining  $L^n := Z^{\zeta_n} / (1 - K^{\zeta_n})$ , the integration-by-parts formula gives  $L^n = 1 + \int_0^{\zeta_n \wedge \cdot} (L_t^n / Z_t) dM_t$ , implying that  $L^n$  is a nonnegative local martingale for all  $n \in \mathbb{N}$ . For  $m \leq n$ , it holds that  $L^m = L^n$  on  $\llbracket 0, \zeta_m \rrbracket$ ; from this consistency property and the nonnegative martingale convergence theorem, it easily follows that there exists a local martingale  $L$  such that  $L = L^n$  holds on  $\llbracket 0, \zeta_n \rrbracket$  for all  $n \in \mathbb{N}$  and  $L_t = \lim_{n \rightarrow \infty} L_{\zeta_n}^n$  holds for all  $t \geq \zeta$ . Since  $K = K^\zeta$ ,  $L = L^\zeta$  and  $Z = Z^\zeta$ , we conclude that  $Z = L(1 - K)$  holds. By the integration-by-parts formula,  $Z = 1 + \int_0^\cdot (1 - K_t) dL_t - \int_0^\cdot L_t dK_t$  holds; comparing with the Doob-Meyer decomposition  $Z = M - A$  of  $Z$ , and recalling that  $A_0 = 0$ , we obtain that  $A = \int_0^\cdot L_t dK_t$ .

**Lemma 2.4.** *Suppose that  $\rho$  is an honest time avoiding all stopping times. Then, with the above notation,  $K_\rho$  has the standard uniform law.*

*Proof.* For  $u \in [0, 1)$ , define the stopping time  $\eta_u := \inf \{t \in \mathbb{R}_+ \mid K_t \geq u\}$ , with the usual convention  $\eta_u = \infty$  if the last set is empty. Since  $K$  has  $\mathbb{P}$ -a.s. continuous paths,  $K_{\eta_u} = u$  holds  $\mathbb{P}$ -a.s. on  $\{\eta_u < \infty\}$  for all  $u \in [0, 1)$ . Recalling that  $A = \int_0^\cdot L_t dK_t$ , a change of variables gives

$$(2.1) \quad \int_0^\infty f(K_t) dA_t = \int_0^\infty f(K_t) L_t dK_t = \int_0^1 L_{\eta_u} \mathbb{I}_{\{\eta_u < \infty\}} f(u) du, \quad \text{for any Borel } f : [0, 1) \mapsto \mathbb{R}_+.$$

The facts that  $Z \leq 1$  and  $(1 - K) \geq 1 - u$  hold up to  $\mathbb{P}$ -evanescence on  $\llbracket 0, \eta_u \rrbracket$  imply that  $\mathbb{P}[L_{\eta_u}^* \leq 1/(1 - u)] = 1$  holds for all  $u \in [0, 1)$ . Therefore,  $\mathbb{E}[L_{\eta_u}] = 1$  holds for all  $u \in [0, 1)$ . Since  $\mathbb{P}[\rho = \infty] = 0$ , it follows that  $\mathbb{P}[Z_\infty = 0] = 1$ ; then,  $\mathbb{P}[Z_\infty = L_\infty(1 - K_\infty)] = 1$  implies  $\mathbb{P}[K_\infty < 1, L_\infty > 0] = 0$ . Therefore, for  $u \in [0, 1)$ , the set-inclusion  $\{\eta_u = \infty\} \subseteq \{K_\infty < 1\}$  implies  $\mathbb{P}[L_{\eta_u} \mathbb{I}_{\{\eta_u < \infty\}} = L_{\eta_u}] = 1$ . Then,  $\mathbb{E}[L_{\eta_u}] = 1$  gives  $\mathbb{E}[L_{\eta_u} \mathbb{I}_{\{\eta_u < \infty\}}] = 1$  for  $u \in [0, 1)$ . By Fubini's Theorem and (2.1), we obtain  $\mathbb{E}[f(K_\rho)] = \mathbb{E}[\int_0^\infty f(K_t) dA_t] = \int_0^1 f(u) du$ . Since the latter holds for any Borel  $f : [0, 1) \mapsto \mathbb{R}_+$ , it follows that  $K_\rho$  has the standard uniform law.  $\square$

Suppose now that  $\rho$  is an honest time avoiding all stopping times, and recall all notation introduced above. Since  $\mathbb{P}[Z_\rho = L_\rho(1 - K_\rho)] = 1$ , Lemma 2.3 gives  $\mathbb{P}[L_\rho = 1/(1 - K_\rho)] = 1$ ; then,  $\mathbb{P}[L_\rho > x] = \mathbb{P}[K_\rho > 1 - 1/x] = 1/x$  for all  $x \in (1, \infty)$  follows from Lemma 2.4. As  $\mathbb{P}[L_\rho \leq L_\infty^*] = 1$ , Lemma 2.1 implies both that  $L \in \mathcal{L}_0^\uparrow$  and that  $\mathbb{P}[L_\rho = L_\infty^*] = 1$ . Then, Lemma 2.2 gives  $L \in \mathcal{L}_0^\uparrow$  and  $\mathbb{P}[\rho = \rho_{\min}^L] = 1$ . Finally, since  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets of  $\mathcal{F}$ , one can alter  $L$  on a set of zero  $\mathbb{P}$ -measure and have that  $\rho = \rho_{\min}^L$  identically holds. This establishes implication (1)  $\Rightarrow$  (2).

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